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Stability Theorems for Impulsive Functional Differential Equations

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Abstract

In this paper, sufficient conditions are derived for asymptotic stability and uniformly asymptotic stability for impulsive functional differential equation using piecewise continuous differential equation.

Keywords: Stability, Impulsive Functional Differential Equation, Liapunov functional

Introduction

Consider the impulsive functional differential equation

$$\begin{cases} x'(t) = f(t, x_t), & t \neq t_k\ t \geq t_0 \\ \Delta x = I_k(t, (x_t^{-})), & t = t_k, k \in Z^+ \end{cases} \quad (1)$$

Where $f: J \times PC \to R^n, \Delta x = x(t) - x(t^-), t_0 < t_1 < \cdots t_k < t_{k+1} < \cdots$, With $t_k \to \infty$ as $k \to \infty$ and $I_k: J \times S(\rho) \to R^n$, where $J = [t_0, \infty)$, $S(\rho) = \{x \in R: |x| < \rho\}. PC = PC([-\tau, 0], R^n)$ denotes the space of piecewise right continuous functions $\varphi: [-\tau, 0] \to R^n$ with sup-norm $\|\varphi\|_\infty = sup_{-\tau \leq s \leq 0}|\varphi(s)|$ and the norm $\|\varphi\|_2 = (\int_{-\tau}^{0}|\varphi(s)|^2 ds)^{1/2,}$ where τ is a positive constant, $\|.\|$ is a norm in $R^n. x_t \in PC$ is defined by $x_t(s) = x(t+s)$ for $-\tau \leq s \leq 0. x'(t)$ denotes the right-hand derivative of $x(t). Z^+$ is the set of all positive integers,

Let $f(t, 0) = 0$ and $J(0) = 0$, then $x(t) = 0$ is the zero solution of (1). Set $PC(\rho) = \{\varphi \in PC: \|\varphi\|_\infty < \rho\}, \forall \rho > 0$.

Definition 1.1

Let σ be the initial time, $\forall\ \sigma \in R$, the zero solution of (1) is said to be

a) stable if , for each $\sigma \geq t_0$ and $\varepsilon > 0$, there is a $\delta = \delta(\sigma, \varepsilon) > 0$ such that , for $\varphi \in PC(\delta)$, a solution $x(t, \sigma, \varphi)$ satisfies $|x(t, \sigma, \varphi)| < \varepsilon$ for $t \geq t_0$.
b) uniformly stable if it is stable and δ in the definition of stability is independent of σ
c) asymptotically stable if it is stable and, for each $t_0 \in R_+$, there is an $\eta = \eta(t_0) > 0$ such that, for $\varphi \in PC(\eta), x(t, \sigma, \varphi) \to 0\ as\ t \to \infty$
d) uniformly asymptotically stable if it is uniformly stable and there is an $\eta > 0$ and , for each $\varepsilon > 0$, a $T = T(\varepsilon) > 0$ such that , for $\varphi \in PC(\eta), |x(t, \sigma, \varphi)| < \varepsilon$ for $t \geq t_0 + T$

Definition 1.2

A functional $V(t, \varphi): J \times PC(\rho) \to R_+$ belong to class $v_o(.)$ (a set of Liapunov like functional) if

a) V is continuous on $[t_{k-1}, t_k) \times PC(\rho)$ for each $k \in Z_+$, and for all $\varphi \in PC(\rho)$ and $k \in Z_+$, the limit $\lim_{(t,\varphi)\to(t_k^-,\varphi)} V(t, \varphi) = V(t_k^-, \varphi)$ exists.
b) V is locally Lipchitzian in φ in each set in $PC(\rho)$ and $V(t, 0) = 0$ The set $\Re$ is defined by $\Re = \{W \in C(R_+, R_+)$: strictly increasing and $W(0) = 0$

Main Results

Theorem 1

Assume that there exist $V_{1,} V_2 \in v_0(.)$, $W_{1,} W_2, W_{3,} W_4 \in \Re$ such that

I. $W_1(\varphi(0)) \leq V(t, \varphi) \leq W_2(\varphi(0))$, where $V(t, \varphi) = V_1(t, \varphi) + V_2(t, \varphi)$
II. $V\big(t_k, x + I_k(t_k, x)\big) - V(t_k^-, x) \leq 0$
III. $a V_1'(t, x_t) + bV_2'(t, x_t) \leq -\lambda(t)\, W_3\, (inf\ \{x(s) : t - h \leq s \leq t\})$
IV. $pV_1'(t, x_t) + qV_2'(t, x_t) \leq 0$

where $a^2 + b^2 \neq 0,\ p^2 + q^2 \neq 0\ and\ \int_0^\infty \lambda(s)\, ds = \infty$

(A) Suppose further that there is a $\mu = \mu(\gamma) > 0$ *for each* $0 < \gamma < H_1$ such that
$$pV_1'(t, x_t) + qV_2'(t, x_t) \le -\mu V_1'(t, x_t) \quad (2)$$
if $x(t) \ge \gamma$. If either $(i)\ a > 0, b > 0$ or $(ii) p \ge 0, q > 0$ hold, then the zero solution of (1) is uniformly and asymptotically stable.

(B) The same is concluded if
$$pV_1'(t, x_t) + qV_2'(t, x_t) \le \mu V_1'(t, x_t)$$
holds in place of (2) and if either $(i)\ a > 0, b > 0$ or $(ii) p > 0, q > 0$.

Proof

We first prove the uniform stability. For given $\varepsilon > 0$, we may choose a $\delta = \delta(\varepsilon) > 0$ such that $W_2(\delta) < W_1(\varepsilon)$. For any $\sigma \ge t_0$ and $\varphi \in PC_\delta$, let $x(t, \sigma, \varphi)$ be the solution of (1). We will prove that
$$x(t, \sigma, \varphi) \le \varepsilon, \qquad t \ge \sigma$$
Let $x(t) = x(t, \sigma, \varphi)$ and $V_1(t) = V_1(t, x_t), V_2(t) = V_2(t, x_t)$ and $V(t) = V(t, x_t)$.

Then by assumption (iv),
$$V'(t, x_t) \le 0, \qquad \sigma \le t_{k-1} \le t < t_k, \quad k \in Z^+$$
and so V(t) is non increasing on the interval of the form $[t_{k-1}, t_k)$. From condition (ii)
$$V(t_k) - V(t_k^-) = V\big(t_k, x(t_k^-) + I_k(t_k, x(t_k^-))\big) - V(t_k^-, x(t_k^-)) \le 0$$
Thus V(t) is non increasing on $[\sigma, \infty)$. We have
$$W_1(x(t)) \le V(t) \le V(\sigma) \le W_2(\sigma) < W_1(\varepsilon), \quad t \ge \sigma$$
This implies with the monotonicity of W_1, $|x(t)| < \varepsilon$ for $t \ge \sigma$ and so that the zero solution of (1) is uniformly stable.

To show asymptotic stability, for a given $t_0 \in R_+$ and a fixed $0 < H_2 < H_1$, take $\eta = \eta(t_0) = \delta(t_0, H_2) > 0$, where δ is that in the definition of stability and for a given $\varphi \in PC(\eta)$, let $x(t) = x(t, \sigma, \varphi)$ be a solution of (1). Suppose for contradiction that $x(t) \nrightarrow 0$ as $t \to \infty$. Then there is a sequence $\{T_i\}$ and an $\varepsilon_0 > 0$ with $T_i \to \infty$ and $|x(T_i)| > \varepsilon_0$. Define $\varepsilon_2 = W_2^{-1}(\frac{W_1(\varepsilon_0)}{2})$ then there is a sequence $\{s_i\}$ with $s_i \to \infty$ and $|x(s_i)| < \varepsilon_2$. Otherwise there is an $S \ge t_0$ such that $|x(t)| \ge \varepsilon_2$ for $t \ge S$ and

$av_1(t) + bv_2(t) \le$
$av_1(S + h) + bv_2(S + h) - \int_{S+h}^{t} \lambda(s) W_4(\inf\{|x(\sigma)|: s - h \le$
σ≤sds+
S+h≤tk≤t[Vtk−S+h≤tk≤t[Vtk−Vtk−)]

$$\le av_1(S + h) + bv_2(S + h) - W_4(\varepsilon_2) \int_S^t \lambda(s) ds \to -\infty$$
as $t \to \infty$, which contradicts either $av_1(t) + bv_2(t) \ge 0$ if (i) holds or
$$av_1(t) + bv_2(t) \ge -|a| W_2(H_2) - |b|(pv_1(t_0) + qV_2(t_0))/q$$
if (ii) holds.

In Case (A), we may assume $T_{i-1} < s_i < T_i$ by choosing and renumbering if necessary. Then we can take a sequence $\{t_i\}$ such that $s_i < t_i < T_i, |x(t_i)| = \varepsilon_2$ and $|x(t)| > \varepsilon_2$ for $t_i < t \le T_i$.

Then $pv_1(T_i) + qv_2(T_i) - \big(pv_1(T_{i-1}) + qv_2(T_{i-1})\big)$
$$\le pv_1(T_i) + qv_2(T_i) - \big(pv_1(t_i) + qv_2(t_i)\big) + \sum_{t_i \le t_k \le T_i} [V(t_k) - V(t_k^-)]$$
$$\le -\mu(\varepsilon_2)\big(v_1(T_i) - v_1(t_i)\big)$$
$$\le -\mu(\varepsilon_2) W_1(\varepsilon_0)/2$$
and a contradiction follows from
$$pv_1(T_n) + qv_2(T_n) = pv_1(T_1) + qv_2(T_1) + \sum_{i=2}^{n} \Big[pv_1(T_i) + qv_2(T_i) - \big(pv_1(T_{i-1}) + qv_2(T_{i-1})\big)\Big] + \sum_{T_{i-1} \le t_k \le T_i} [V(t_k) - V(t_k^-)]$$
$$\le pv_1(T_1) + qv_2(T_1) - \frac{(n-1)\mu(\varepsilon_2) W_1(\varepsilon_0)}{2} \to -\infty$$
as $n \to \infty$

In Case (B), we may assume $s_{i-1} < T_i < s_i$ and take $\{t_i\}$ with $T_i < t_i < s_i$, $|x(t_i)| = \varepsilon_2$ and $|x(t)| > \varepsilon_2$ for $T_i \le t < t_i$ so that
$$pv_1(t_i) + qv_2(t_i) - \big(pv_1(t_{i-1}) + qv_2(t_{i-1})\big)$$
$$\le pv_1(t_i) + qv_2(t_i) - \big(pv_1(T_i) + qv_2(T_i)\big) + \sum_{T_i \le t_k \le t_i} [V(t_k) - V(t_k^-)]$$
$$\le \mu(\varepsilon_2)(v_1(t_i) - v_1(T_i))$$
$$\le -\mu(\varepsilon_2) W_1(\varepsilon_0)/2$$
This implies a contradiction by the same argument as in case (A)

Therefore, $x(t) \to 0$ as $t \to \infty$. The proof is complete.

Theorem 2.

Assume that there exist $V_1, V_2 \in v_0(.)$ and $W_1, W_2, W_3, W_4 \in \Re$ such that

a) $W_1|\varphi(0)| \le V(t, \varphi) \le W_2|\varphi(0)|$ where $V(t, \varphi) = V_1(t, \varphi) + V_2(t, \varphi)$
b) $V\big(t_k, x + I_k(t_k, x)\big) - V(t_k^-, x) \le 0$, $k \in Z^+$
c) $aV'_1(t, x_t) + bV'_2(t, x_t) \le -\lambda(t) W_3(\inf\{|x(s)|; t - h \le s \le t\})$ *and* $pV'_1(t, x_t) + qV'_2(t, x_t) \le 0$

Where $a^2 + b^2 \ne 0, p^2 + q^2 \ne 0$ *and*
$$\lim_{S \to \infty} \int_t^{t+S} \lambda(s) ds = \infty \text{ uniformly in } t \in R_+$$

A. Suppose that there is a $\mu = \mu(\gamma) > 0$ for each $0 < \gamma < H_1$ such that
$$pV'_1(t, x_t) + qV'_2(t, x_t) \le -\mu V'_1(t, x_t) \quad (3)$$
If $|x(t)| \ge \gamma$. If either (i) $a > 0, b \ge 0$ or (ii) $p \ge 0, q \ge 0$ hold, then the zero solution of (1) is uniformly asymptotically stable.

B. The same is concluded if (3) is replaced by
$$pV'_1(t, x_t) + qV'_2 \le \mu V'_1(t, x_t)$$

And if either (i) $a > 0, b \ge 0$ or (ii) $p > 0, q \ge 0$ hold

Proof

Uniform Stability can be proven as stability in Theorem 1.

Set $\eta = \delta(H_2)$ for a fixed $0 < H_2 < H_1$ and δ in the definition of uniform stability. For given $t_0 \in R_+, \varphi \in C_\eta$, let $x(t) = x(t,\sigma,\varphi)$ be a solution of (1). Let $\varepsilon > 0$ be given and take $\delta = \delta(\varepsilon) > 0$ of uniform stability. Define $\delta_1 = W_2{}^{-1}(\frac{W_1(\delta)}{2})$. Choose a $S = S(\varepsilon) > 0$ with

$$\int_t^{t+S} \lambda(s)ds > 2(|a|W_2(H_2) + |b|W_3(H_2))/W_4(\delta_1)$$

For $t \in R_+$ and an integer $N = N(\varepsilon) \geq 1$ with $N\mu(\delta_1)W_1(\delta)/2 > 2(|p|W_2(H_2) + |q|W_3(H_2))$

Define $T = T(\varepsilon) = N(S + 2h)$. Suppose, for contradiction, that $||x_t|| \geq \delta$ for $t_0 \leq t \leq t_0 + T$.

In Case (A), for $1 \leq i \leq N$, there is a

$$+(i-1)(S+2h) \leq s_i \leq t_0 + (i-1)(S+2h) + h + S$$

With $|x(s_i)| < \delta_1$.Otherwise $|x(t)| \geq \delta_1$ on this interval and, for $I_i = [t_0 + (i-1)(S+2h) + h, t_0 + (i-1)(S+$ *2h+h+S, v1t=V1(t,xt)* and *v2t=V2(t,xt)*, we have

$$-2(|a|W_2(H_2) + |b|W_3(H_2)) \leq av_1(t_0 + (i-1)(S+2h) + h + S) + bv_2(t_0 + (i-1)(S+2h) + h + S)$$

$(-av_1(t_0 + (i-1)(S+2h) + h) + bv_2(t_0 + (i-1)(S+$ *2h+h))*

$$\leq -\int \lambda(t)W_4(\inf\{|x(s)|: t-h \leq s \leq t\})ds$$
$$\leq -W_4(\delta_1)\int \lambda(t) \ < \ -2(|a|W_2(H_2) + |b|W_3(H_2))$$

This inequality also holds true as per condition (ii)
a contradiction.

From the supposition , for $1 \leq i \leq N$, there is a

$$t_0 + (i-1)(S+2h) + h + S \leq T_i \leq t_0 + i(S+2h)$$

Such that $|x(T_i)| \geq \delta$. Thus, there is an $s_i < t_i < T_i$ with $|x(t_i)| = \delta_1$ and $|x(t)| > \delta_1$ for $t_i < t \leq T_i$. We obtain

$$pv_1\big(t_0 + i(S+2h)\big) + qv_2\big(t_0 + i(S+2h)\big) - \big(pv_1\big(t_0 + (i-1)(S+2h)\big) + qv_2(t_0 + (i-1)(S+2h))\big)$$
$$\leq pv_1(T_i) + qv_2(T_i) - (pv_1(t_i) + qv_2(t_i))$$
$$\leq -\mu(\delta_1)\big(v_1(T_i) - v_1(t_i)\big) \ \leq \ -\mu(\delta_1)W_1(\delta)/2$$

And

$-2\big(|p|W_2(H_2) + |q|W_3(H_2)\big) \leq \ pv_1\big(t_0 + N(S+2h)\big) + qv_2\big(t_0 + N(S+2h)\big) - (pv_1(t_0) + q(v_2(t_0))$

$=\sum_{i=1}^{N}(pv_1\big(t_0 + i(S+2h)\big) + qv_2\big(t_0 + i(S+2h)\big)) -$ *(pv1t0+i−1S+2h+qv2t0+i−1S+2h)*

$$\leq -N\mu(\delta_1)W_1(\delta)/2 \ < \ -2(|p|W_2(H_2) + |q|W_3(H_2)),$$

This inequality also holds true as per condition (ii)
a contradiction.

In Case (B), we can take, for $1 \leq i \leq N$, $t_0 + (i-$ *12h+S+h≤si≤t0+i2h+S* with */x(si)/<δ1*, $t_0 + (i-1)(2h+S) \leq T_i \leq t_0 + (i-1)(2h+S) + h$ with $|x(T_i)| \geq \delta$ and $T_i < t_i < s_i$ with $|x(t_i)| = \delta_1$, $|x(t)| > \delta_1$ for $T_i \leq t < t_i$ so that

$$pv_1\big(t_0 + i(S+2h)\big) + qv_2\big(t_0 + i(S+2h)\big) - \big(pv_1\big(t_0 + (i-1)(S+2h)\big) + qv_2(t_0 + (i-1)(S+2h))\big)$$
$$\leq pv_1(t_i) + qv_2(t_i) - (pv_1(T_i) + qv_2(T_i))$$
$$\leq \mu(\delta_1)\big(v_1(t_i) - v_1(T_i)\big) \ \leq \ -\mu(\delta_1)W_1(\delta)/2$$

This inequality also holds true as per condition (ii)
a contradiction follows from this as in case(A)

Consequently $||x_{t'}|| < \delta$ for some $t_0 \leq t' \leq t_0 + T$ and $|x(t)| < \varepsilon$ for $t \geq t_0 + T$. This completes the proof.

Corollary

If there are $V_1, V_2 \in v_0(.)$ and $W_1, W_2, W_3, W_4 \in \Re$ satisfying

a) $W_1|\varphi(0)| \leq V(t,\varphi) \leq W_2|\varphi(0)|$

b) $0 \leq V(t,\varphi) \leq W_3(\big||\varphi|\big|)$ *where* $V(t,\varphi) = V_1(t,\varphi) + V_2(t,\varphi)$

c) $V\big(t_k, x + I_k(t_k,x)\big) - V(t_k{}^-, x) \leq 0$

d) $V'_1(t,x_t) + c_1V'_2(t,x_t) \leq 0$

e) $V'_1(t,x_t) + c_2V'_2(t,x_t) \leq -\lambda(t)W_4(inf\{|x(s)|; t-h \leq s \leq t\})$

Where $c_1 \neq c_2$ either $c_1 \geq 0$ or $c_2 \geq 0$ and $lim_{S\to\infty}\int_t^{t+S}\lambda(s)ds = \infty$ uniformly in $t \in R_+$

Then the zero solution of (1) is uniformly asymptotically stable.

Proof

We may assume that $c_1 > c_2$. Then $c_1 \geq 0$, if $c_2 = 0$

$$V'_1(t,x_t) + c_1V'_2(t,x_t) \leq 0 \leq -V'_1(t,x_t)$$

And the conditions of theorem 2(A ii) are satisfied.

If $c_1 > 0$

$$V'_1(t,x_t) + c_1V'_2(t,x_t) \leq (c_1 - c_2)V'_2(t,x_t) \leq -(\frac{(c_1-c_2)}{c_1})V'_1(t,x_t)$$

Implies uniform stability by Theorem 2(A ii).

Example Consider the impulsive differential equation

$$x'(t) = -a(t)f\big(x(t)\big) + b(t)g\big(x(t-h)\big)$$
$$x(t_k) - x(t_k{}^-) = c_kx(t_k{}^-), \ \ k \in Z^+$$

Where $a: R_+ \to R_+$, $b: R_+ \to R$, $f, g: R \to R$ are continuous, $xf(x) > 0$, for $x \neq 0, |g(x)| \leq c|f(x)|$ for $c > 0$ and $g(x) \neq 0$ for $x \neq 0$, $|1+c_k| \leq 1, k \in Z^+$ and $\sum_{k=1}^{\infty}[1 - |1 + c_k|] = \infty$

If $\int_t^{t+h}|b(s)|ds$ is bounded, $a(t) - \alpha c|b(t+h)| \geq 0$

For some $\alpha > 1$, and for some $1 \leq \beta \leq \alpha$, $\lambda(t) = a(t) - \beta c|b(t+h)| + (\beta - 1)|b(t)|$ satisfies

$$\lim_{S\to\infty}\int_t^{t+S}\lambda(s)ds = \infty$$

Uniformly in $t \in R_+$, then the zero solution is uniformly asymptotically stable.

Proof

Let $V = V_1 + V_2$ where $V_1(t,\varphi) = |\varphi(0)|$, $V_2(t,\varphi) = \int_{-h}^{0}|b(t+s+h)||g(\varphi(s)|\,ds$

Then $V_2(t,\varphi) \leq \int_t^{t+h}|b(s)|dsW_3(||\varphi||)$ for some function $W_3 \in \Re$

And $V_1(t_k, x + c_kx) - V_1(t_k{}^-, x) = |(1+c_k)x| - |x| = [1 - |1 + c_k|]V(t_k{}^-, x)$

Let $\lambda_k = 1 - |1 + c_k|$; then $\sum_{k=1}^{\infty}\lambda_k = \infty$. We check that for any $\alpha > 0$, there is a $\beta > 0$ such that $V(t,x_t) \geq \alpha$ implies $V_1(t,x_t) \geq \beta$.

Otherwise we must have $\lim\inf_{t\to\infty}V_1(t,x_t) = 0$

We let $V(t) = V_1(t,x_t) + V_2(t,x_t)$

Then $V(t_k) - V(t_k{}^-) = V_1\big(t_k, x(t_k{}^-) + c_kx(t_k{}^-)\big) - V_1\big(t_k{}^-, x(t_k{}^-)\big) \leq 0$

$$V'_1(t,x_t) + \beta V'_2(t,x_t) \leq -(a(t) - \beta c|b(t+h)|)\big|f\big(x(t)\big)\big| - (\beta - 1)|b(t)|\big|g\big(x(t-h)\big)\big| + \sum_{0\leq t_k\leq t} V(t_k) - V(t_k{}^-)$$
$$\leq -\lambda(t)W_4(\inf\{|x(s)|: t-h \leq s \leq t\})$$

If $||x_t|| \leq H$ for a fixed $0 < H < \infty$ and some function W_4.

If $\beta = 1$, for $\alpha \neq 1$ $V'_1(t,x_t) + \alpha V'_2(t,x_t) \leq 0$

If $\beta > 1$ $\quad V'_1(t, x_t) + 1\, V'_2(t, x_t) \leq 0$

The conditions of the corollary are satisfied and hence the zero solution is uniformly asymptotically stable.

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